Ejercicios Teoría Cuántica de Campos. Capítulo 53

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4 de marzo de 2021

1. Calcular $\left[a_p^\dagger a_n, a_n^\dagger a_p\right]$.

Conociendo las relaciones

$$a_i a_j^\dagger + a_j^\dagger a_i = \delta_{ij}, \qquad a_i^\dagger a_j^\dagger + a_j^\dagger a_i^\dagger = 0, \qquad a_i a_j + a_j a_i = 0 \tag{1}$$

Podemos usarlas en la forma $a_n a_n^\dagger = 1 - a_n^\dagger a_n$, $a_p a_p^\dagger = 1 - a_p^\dagger a_p$ y $a_p^\dagger a_n^\dagger = -a_n^\dagger a_p^\dagger$, $a_n a_p = -a_p a_n$.

$$\begin{aligned}\left[a_p^\dagger a_n, a_n^\dagger a_p\right] &= a_p^\dagger a_n a_n^\dagger a_p - a_n^\dagger a_p a_p^\dagger a_n = a_p^\dagger(1 - a_n^\dagger a_n) a_p - a_n^\dagger(1 - a_p^\dagger a_p) a_n \\ &= a_p^\dagger a_p - a_p^\dagger a_n^\dagger a_n a_p - a_n^\dagger a_n + a_n^\dagger a_p^\dagger a_p a_n = a_p^\dagger a_p - \cancel{a_n^\dagger a_p^\dagger a_p a_n} - a_n^\dagger a_n + \cancel{a_n^\dagger a_p^\dagger a_p a_n} \\ &= a_p^\dagger a_p - a_n^\dagger a_n\end{aligned}$$

2. Comprobar las relaciones de conmutación $[I_i, I_j] = i\varepsilon_{ijk} I_k$.

Con los operadores de Isospín definidos como

$$I_1 = \frac{1}{2}\left(a_p^\dagger a_n + a_n^\dagger a_p\right), \qquad I_2 = \frac{i}{2}\left(a_n^\dagger a_p - a_p^\dagger a_n\right), \qquad I_3 = \frac{1}{2}\left(a_p^\dagger a_p - a_n^\dagger a_n\right)$$

$$[I_1, I_2] = \frac{i}{4}\left(\left[a_p^\dagger a_n, a_n^\dagger a_p\right] - \left[a_n^\dagger a_p, a_p^\dagger a_n\right]\right) = \frac{i}{2}\left[a_p^\dagger a_n, a_n^\dagger a_p\right] \tag{2}$$

$$[I_2, I_3] = \frac{i}{4}\left(\left[a_n^\dagger a_p, a_p^\dagger a_p\right] - \left[a_n^\dagger a_p, a_n^\dagger a_n\right] - \left[a_p^\dagger a_n, a_p^\dagger a_p\right] + \left[a_p^\dagger a_n, a_n^\dagger a_n\right]\right) \tag{3}$$

$$[I_3, I_1] = -\frac{1}{4}\left(\left[a_n^\dagger a_n, a_p^\dagger a_n\right] + \left[a_n^\dagger a_n, a_n^\dagger a_p\right] - \left[a_p^\dagger a_p, a_p^\dagger a_n\right] - \left[a_p^\dagger a_p, a_n^\dagger a_p\right]\right) \tag{4}$$

El primer conmutador ya lo hemos calculado

$$\left[a_p^\dagger a_n, a_n^\dagger a_p\right] = a_p^\dagger a_p - a_n^\dagger a_n$$

Por lo que obtenemos directamente

$$[I_1, I_2] = \frac{i}{2}\left(a_p^\dagger a_p - a_n^\dagger a_n\right) = iI_3 \tag{5}$$

Para los otros conmutadores primero debemos calcular los siguientes:

$$\left[a_p^\dagger a_p, a_p^\dagger a_n\right] = a_p^\dagger a_p a_p^\dagger a_n - a_p^\dagger a_n a_p^\dagger a_p = a_p^\dagger a_p a_p^\dagger a_n - \cancelto{0}{a_p^\dagger a_p^\dagger} a_p a_n = a_p^\dagger(1 - a_p^\dagger a_p) a_n = a_p^\dagger a_n - \cancelto{0}{a_p^\dagger a_p^\dagger} a_p a_n \tag{6}$$

$$\left[a_p^\dagger a_p, a_n^\dagger a_p\right] = a_p^\dagger a_p a_n^\dagger a_p - a_n^\dagger a_p a_p^\dagger a_p = -a_n^\dagger a_p a_p^\dagger a_p = -a_n^\dagger a_p(1 - a_p a_p^\dagger) = -a_n^\dagger a_p \tag{7}$$

$$\left[a_n^\dagger a_n, a_p^\dagger a_n\right] = a_n^\dagger a_n a_p^\dagger a_n - a_p^\dagger a_n a_n^\dagger a_n = -a_p^\dagger a_n(1 - a_n a_n^\dagger) = -a_p^\dagger a_n \tag{8}$$

$$\left[a_n^\dagger a_n, a_n^\dagger a_p\right] = a_n^\dagger a_n a_n^\dagger a_p - a_n^\dagger a_p a_n^\dagger a_n = (1 - a_n a_n^\dagger) a_n^\dagger a_p = a_n^\dagger a_p \tag{9}$$

Sustituyendo en los conmutadores de Isospín obtenemos finalmente

$$[I_2, I_3] = \frac{i}{4}\left(a_n^\dagger a_p + a_n^\dagger a_p + a_p^\dagger a_n + a_p^\dagger a_n\right) = \frac{i}{2}\left(a_n^\dagger a_p + a_p^\dagger a_n\right) = iI_1 \tag{10}$$

$$[I_3, I_1] = -\frac{1}{4}\left(-a_p^\dagger a_n + a_n^\dagger a_p - a_p^\dagger a_n + a_n^\dagger a_p\right) = i\frac{i}{2}\left(a_n^\dagger a_p - a_p^\dagger a_n\right) = iI_2 \tag{11}$$

3. Invertir las siguientes relaciones

$$
\begin{aligned}
\left|\Delta^{++}\right\rangle &= \left|\pi^{+}, p\right\rangle \\
\left|\Delta^{+}\right\rangle &= \frac{1}{\sqrt{3}}\left|\pi^{+}, n\right\rangle + \sqrt{\frac{2}{3}}\left|\pi^{0}, p\right\rangle \\
\left|\Delta^{0}\right\rangle &= \sqrt{\frac{2}{3}}\left|\pi^{0}, n\right\rangle + \frac{1}{\sqrt{3}}\left|\pi^{-}, p\right\rangle \\
\left|\Delta^{-}\right\rangle &= \left|\pi^{-}, n\right\rangle \\
\left|N^{+}\right\rangle &= \frac{1}{\sqrt{3}}\left|\pi^{0}, p\right\rangle - \sqrt{\frac{2}{3}}\left|\pi^{+}, n\right\rangle \\
\left|N^{0}\right\rangle &= \sqrt{\frac{2}{3}}\left|\pi^{-}, p\right\rangle - \frac{1}{\sqrt{3}}\left|\pi^{0}, n\right\rangle
\end{aligned}
$$

Estas relaciones, escogiendo las siguientes bases:

$$
\mathscr{B}_1 = \{\left|\Delta^{++}\right\rangle, \left|\Delta^{+}\right\rangle, \left|\Delta^{0}\right\rangle, \left|\Delta^{-}\right\rangle, \left|N^{+}\right\rangle, \left|N^{0}\right\rangle\}
$$

$$
\mathscr{B}_2 = \{\left|\pi^{+}, p\right\rangle, \left|\pi^{+}, n\right\rangle, \left|\pi^{0}, p\right\rangle, \left|\pi^{0}, n\right\rangle, \left|\pi^{-}, p\right\rangle, \left|\pi^{-}, n\right\rangle\}
$$

Se pueden resumir con la matriz de cambio de base

$$
M(\mathscr{B}_1 \leftarrow \mathscr{B}_2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0
\end{pmatrix} \tag{12}
$$

Esta matriz, al relacionar dos bases ortonormales, debe ser una matriz ortogonal. Es decir que $M^{-1} = M^{t}$. Esto implica que la matriz de cambio de base inversa será:

$$
M(\mathscr{B}_2 \leftarrow \mathscr{B}_1) = M(\mathscr{B}_1 \leftarrow \mathscr{B}_2)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 & 0 & -\sqrt{\frac{2}{3}} & 0 \\
0 & \sqrt{\frac{2}{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \sqrt{\frac{2}{3}} & 0 & 0 & \frac{-1}{\sqrt{3}} \\
0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix} \tag{13}
$$

O, equivalentemente:

$$
\begin{aligned}
\left|\pi^{+}, p\right\rangle &= \left|\Delta^{++}\right\rangle \\
\left|\pi^{+}, n\right\rangle &= \frac{1}{\sqrt{3}}\left|\Delta^{+}\right\rangle - \sqrt{\frac{2}{3}}\left|N^{+}\right\rangle \\
\left|\pi^{0}, p\right\rangle &= \sqrt{\frac{2}{3}}\left|\Delta^{+}\right\rangle + \frac{1}{\sqrt{3}}\left|N^{+}\right\rangle \\
\left|\pi^{0}, n\right\rangle &= \sqrt{\frac{2}{3}}\left|\Delta^{0}\right\rangle - \frac{1}{\sqrt{3}}\left|N^{0}\right\rangle \\
\left|\pi^{-}, p\right\rangle &= \frac{1}{\sqrt{3}}\left|\Delta^{0}\right\rangle + \sqrt{\frac{2}{3}}\left|N^{0}\right\rangle \\
\left|\pi^{-}, n\right\rangle &= \left|\Delta^{-}\right\rangle
\end{aligned}
$$

4. **Calcular** $\sigma(\pi^- p \to \pi^- p)$

$$\sigma(\pi^- p \to \pi^- p) \sim \left|\langle \pi^- p | S | \pi^- p\rangle\right|^2 = \left| \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix} \right|^2$$

$$= \left| \frac{\alpha}{3} + \frac{2\beta}{3} \right|^2 = \frac{1}{9} \left|\alpha + 2\beta\right|^2$$